Functional Analysis 2020-2021

Syllabus

1. **Linear operators:**
- Bounded linear operators, adjoints and their properties. Projections in Hilbert spaces, Neumann series.
- Compact operators: definition, examples and main properties (composition, closedness, adjoint).
- Fredholm theory: statement and proof.

2. **Spectral theory:**
- Definition and topological properties of the spectrum, decomposition of the spectrum, examples (Volterra, right and left shift, multiplication operators on sequences spaces).
- Spectrum of selfadjoint operators: reality, Weyl 's criterion, extremal of the spectrum.
- Spectrum of compact operators: main properties, spectral theorem for compact selfadjoint operators, variational method to compute eigenvalues, application to integral operators.

3. **Spectral theorem:**
- Continuous functional calculus: spectral mapping theorem, isometric property, proof of the spectral theorem, applications (norm of the resolvent, square root).
- Borellian functional calculus: the spectral measure, construction of bounded Borel functional calculus and its properties.
- Spectral theorem for bounded linear selfadjoint operators: projection valued measure and their property, spectral intergrals and its functional calculus properties, proof of the spectral theorem.
- Applications: characterization of the spectrum, stability of the essential spectrum, *projectors and stability of isolated eigenvalues.

4. **Sturm-Liouville problems**
- Weak formulation of Sturm-Liouville problems; construction of weak solutions via Lax-Milgram in case of small potential; compactness of the solution map; extension to large potentials via Fredholm theory; return to classical solutions; Neumann boundary problem.
- Spectral analysis of Sturm-Liouville problems.
- *Evolution problems, *problems on the line.

5. **Differential calculus in Banach spaces**
- Frechet , Gateux derivatives and their relations; higher order differentiability and Taylor formula.

- Nemitski operator: continuity and differentiability over continuous and L^p integrable spaces.
- Implicit function theorem in Banach spaces, inverse function theorem.
- Applications to semilinear Sturm-Liouville problem and to continuity of the solution map of ODEs with respect to the initial datum.
- Lagrange multipliers: Hilbert and Banach case, complementary spaces, application to Sturm-Liouville problems.

6. Introduction to bifurcation theory

- Lyapunov-Schmidt reduction.
- Crandall-Rabinowitz theorem of bifurcation from simple eigenvalues.
- Applications to nonlinear Sturm-Liouville problems.
- *Stokes wave for water waves.

7. Introduction to topological degree theory

- Construction of Brouwer degree for regular points and C^1 functions, Brouwer fixed point theorem.
- Construction of Leray-Schauder degree, Shauder fixed point theorem.

*The topics marked with * are optional*